Pergamon

Int. J. Solids Structures Vol. 35, No. 18, pp. 2363–2378, 1998

0020–7683/98 $19.00 + .00

PII : S0020–7683(97)00155–8

THE GENERALIZED CATTANEO PARTIAL SLIP PLANE CONTACT PROBLEM. II—EXAMPLES†

MICHELE CIAVARELLA‡
Department of Engineering Science, Oxford University, Parks Road, Oxford OX1 3PJ, U.K.

(*Received* 27 *March* 1997; *in revised form* 28 *May* 1997)

Abstract—This second part of the paper uses the method devised in the part I to give explicit solution to several cases of Cattaneo's plane contact problem, where a monotonically increasing tangential load, starting from zero, is applied to the bodies in contact, with normal loading held fixed. The method consists in reducing the partial slip problem to a superposition of frictionless normal contact problems, for which several results are available, including some recent cases studied by the author. Therefore, a comprehensive set of results is given for single, multiple and periodical contacts. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

In part I of the paper we have given a general method for solving Cattaneo's plane contact problem, using an analogy with an appropriate frictionless normal contact problem. Then, several general solutions in quadrature, for the single, multiple and periodic contact area were given.

In case the geometry of the contact is described by simple functions, the quadrature becomes particularly simple. Indeed, there are many closed form solutions for the traction distributions arising at the contact interface (whereas the complete solution of the contact problem, i.e. including the interior stress field, presents in general greater difficulty), especially for the case of the single contact area. However, even for this simplest case, apart from the classical cases of the parabola, the flat and the wedge indenters, the solutions are not easily available in the literature, and are not worked out explicitly. It is possible, instead, to achieve a greater understanding of the parameters affecting the actual distribution of tractions and, therefore, infer the characteristics of the transmission of loads, and so the strength or fracture mechanics of the contact, by working out solutions for more general configurations. In recent papers, for example, the author has treated cases of profile not defined by a single function, such as a wedge with rounded tip (Ciavarella *et al.*, 1997a, b), or a flat punch with rounded corners (Ciavarella *et al.*, 1997a, b), and the same technique has been applied to a general symmetrical spline profile (Ciavarella *et al.*, 1997c), i.e. piecewise continuous or discontinuous§ linear or quadratic, profile.

Regarding multiple contacts, moreover, many results are little known (like many solutions given in the book by Scthayerman, 1949). For the multiple contact problem, in particular, closed form solutions are possible at least for the simplest cases of two parabolic punches (Gladwell, 1980), or two flat punches (Scthayerman, 1949).

Finally, for periodic contacts, the solution in many cases is also possible in closed form, at least for sinusoidal, squared sinusoidal or periodically flat indenter (Scthayerman, 1949). We, therefore, collect these solutions, and give the explicit transformation in term of Cattaneo's problem, showing particular properties of the single cases. Apart from calculating actual traction distributions, we will particularly concentrate on the much easier calculation of the relation between tangential load Q, and the size of the stick zone c. In general, here only the basic results are summarized, whereas the original papers should be consulted for elucidation of some of the contact laws that can be obtained in closed form.

† Transmitted by D. A. Hills.
‡ Permanent address: Dipartimento di Progettazione e Produzione Industriale, Politecnico di Bari, Viale Japigia 182, 70126 Bari, Italy.
§ In which cases, logarithmic singularities are predicted in general in the discontinuities.

2. SINGLE CONTACT AREAS

We refer to the notation, and the results given in part I of the paper. In particular, eqn (I-20)† gives the integral equation for obtaining the solution of the partial slip problem, in terms of the corrective shear $q^*(\xi)$, whereas for the particular case of single contact area, eqns (I-32) and (33) give the solution in quadrature. We specialize here these latter results for some particular cases of geometry. Where not mentioned differently, we indicate with c the half-width of the stick zone, whereas with a we indicate the half-width of the contact zone.

2.1. *Cylinder*

As this case is so well known, we will just summarize the main results, using the general procedure. For this geometry the standard Hertzian approximation is $h'(x) = -kx$, where k is the reciprocal of the radius of curvature. The stick zone is centrally positioned and is of half-width c. The dimensionless normalized shearing force (Q/fP), as a function of the size of the stick zone (c/a), is found immediately from eqn (I-33), as

$$\frac{Q}{fP} = 1 - \left(\frac{c}{a}\right)^2. \tag{1}$$

The solution for the tractions then reads, from eqn (I-32)

$$p(x) = -\frac{k}{\pi A}\sqrt{a^2-x^2}\int_{-a}^{a}\frac{t}{\sqrt{a^2-t^2}(x-t)}\,\mathrm{d}t = -\frac{k}{A}\sqrt{a^2-x^2} \tag{2}$$

$$-q^*(x)/f = -\frac{k}{\pi A}\sqrt{c^2-x^2}\int_{-c}^{c}\frac{t}{\sqrt{c^2-t^2}(x-t)}\,\mathrm{d}t = -\frac{k}{A}\sqrt{c^2-x^2} \tag{3}$$

$p(x)$ and $q^*(\xi)$ have the well known elliptical shape. It is perhaps interesting to remark that the first problem to be solved for the cylinder, somehow more complicated, was the rolling problem considered by Carter (1926) well before the Cattaneo (1938) and Mindlin (1949) analysis, even though the latter analysis are for general 3-D contacts. The Cattaneo problem for the cylinder is not directly considered by Cataneo and Mindlin.

2.2. *Flat punch*

We have already shown that a partial slip solution cannot be predicted, for $\beta = 0$, for this problem. If $|Q| < fP$ stick persists everywhere, whilst when $|Q| = fP$ slip spontaneously envelopes the entire contact. However, the result merits more discussion, and we first re-work the problem assuming that an explicit solution is possible with a simple formulation. For $\beta = 0$, two cases are possible: the punch is rigid, and the half-plane is incompressible ($\nu = 0.5$), or the punch is elastic, and the half-plane has elastic constants to satisfy exactly the condition $\beta = 0$. In the former case, the contact pressure distribution is then explicitly and exactly given by (Hills *et al.*, 1993, Section 2.8)

$$p(x) = -\frac{P}{\pi\sqrt{a^2-x^2}}. \tag{4}$$

In the latter case, i.e. for the contact of two elastically similar half-planes, it must be borne in mind that, if there is not enough support of material in the region adjacent to the contact area (for example, if the punch is rectangular in shape), the half-plane assumption can be questioned. However, as for the normal pressure distribution the main difference

† In the following we use the notation eqn (I-20) for eqn (20) of part I of the paper, and so on.

will be only in the asymptotics,† i.e. in the region very close to the edges, a partial slip condition, if one exists, has to be limited to the case of a very small region of slip, therefore, not altering much the indicated result. From a different point of view, apart from the mathematical difficulty, the effect is very likely to be comparable with geometrical differences from the perfectly-flat idealization, including imperfections of manufacture. There is, therefore, *no predictable* partial slip solution, based on the initial geometry, and the assumptions on the elasticity of the punch and half-plane.

2.3. *Wedge, power-law and polynomial punches*

Consider next a wedge-shaped punch having a profile such that $h(x) = T_y - \alpha|x|$. Regarding the relation between tangential load and size of the stick area, on putting $t = c\sin\theta$, one has

$$\int_{-c}^{c} \frac{h'(t)t\,dt}{\sqrt{c^2-t^2}} = 2\int_{0}^{c} \frac{t\,dt}{\sqrt{c^2-t^2}} = 2c\cos\theta|_{\pi/2}^{0} = 2c \tag{5}$$

so that, from eqn (I-33)

$$\frac{Q}{fP} = 1 - \frac{c}{a}. \tag{6}$$

The tractions are given by

$$p(x) = -\frac{\alpha}{\pi A}\sqrt{a^2-x^2}\int_{-a}^{a} \frac{|t|}{\sqrt{a^2-t^2}(x-t)}\,dt = -\frac{2\alpha}{\pi A}\cosh^{-1}\left|\frac{a}{x}\right| \tag{7}$$

$$-q^*(x)/f = -\frac{\alpha}{\pi A}\sqrt{c^2-x^2}\int_{-c}^{c} \frac{|t|}{\sqrt{c^2-t^2}(x-t)}\,dt = -\frac{2\alpha}{\pi A}\cosh^{-1}\left|\frac{c}{x}\right| \tag{8}$$

and it is possible also to work out the explicit displacement fields (Truman *et al.*, 1995).

In general, for a punch of profile x^k (which includes polynomials of any order), one obtains (see Appendix I for details)

$$\frac{Q}{fP} = 1 - \left(\frac{c}{a}\right)^k \tag{9}$$

which is a very simple result, which could have been anticipated from self-similarity considerations. Figure 1 summarizes the results obtained for punches of profile $k = 1, 2, 4, 6$, i.e. wedge, parabolic (Hertzian), and higher-order polynomials. It is interesting to note again that for a wedge-punch the size of the slip zones varies linearly with the tangential force transmitted, whilst for a high order punch the slip zone size varies weakly with light shear loads, strongly with high shear loads. From a different point of view, this means that a high order polynomial punch is hardly far from either full-stick or full-sliding conditions, which is consistent with the limiting result that for a flat punch there is no predictable partial slip condition, i.e. it has a 'on-off' behaviour in terms of full-stick, full slip. The full solution in terms of contact pressure (and, therefore, also the shearing traction in the partial slip regime), for any value of k, is reported in Appendix I.

† In particular, the correct singularity predicted by infinitesimal linear theory of elasticity is lower than the classical inverse square root. Also, it is worth remarking that for an external angle lower than a certain threshold, the power-root singularity disappears, and only the logarithmic singularity, already predicted in half-plane elasticity, is present. However, the difference is limited to a very small region close to the edge.

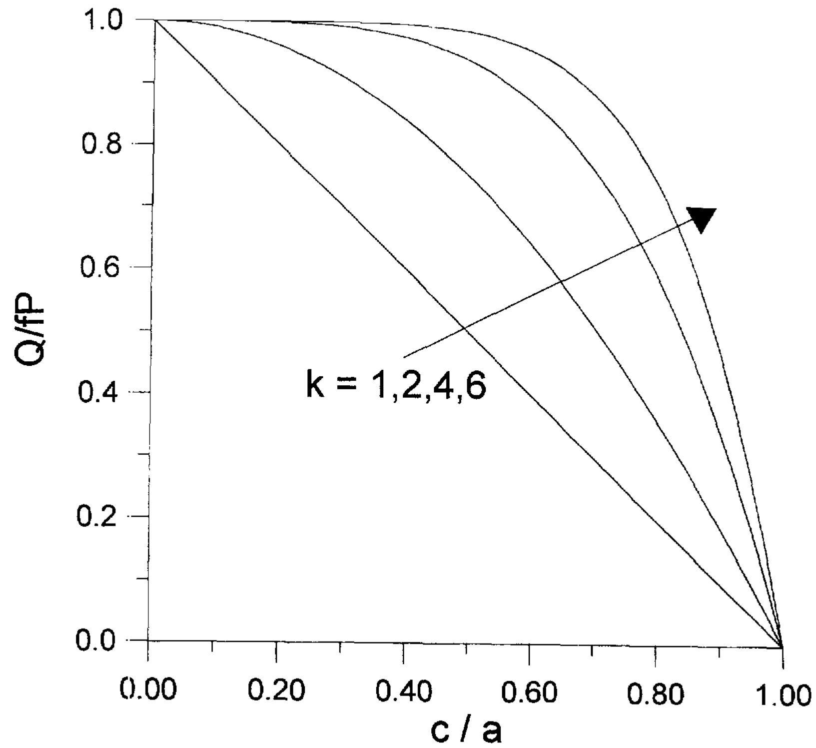

Fig. 1. Relationship $Q/fP - c/a$ for power-law profiles $k = 1, 2, 4, 6$. The case $k = 1$ is a sharp wedge indenter, $k = 2$ is the parabolic Hertzian indenter.

2.4. *Wedge with rounded apex*

Here we summarize results given in full detail in the paper (Ciavarella *et al.*, 1997a, b), to which the reader is referred for the derivation of the results, together with a complete treatment of the strength of the contact. Consider a wedge-shaped punch, with rounded tip, in contact with the half-plane. The function $h'(x)$ is defined by

$$h'(x) = \begin{cases} \alpha, & -a \leqslant x \leqslant -b \\ -\alpha(x/b), & -b \leqslant x \leqslant +b \\ -\alpha, & +b \leqslant x \leqslant +a \end{cases} \tag{10}$$

where α is the external angle of the wedge, which has to be small if the half-plane assumption is to be justified, a is the half-width of the contact area, and b is the half-width of the rounded part. Now, on substituting $t = a\sin\theta$, and defining $b = a\sin\theta_0$:

$$\begin{aligned} \int_{-a}^{a} \frac{h'(t)t\,\mathrm{d}t}{\sqrt{a^2-t^2}} &= \alpha\int_{-a}^{-b}\frac{t\,\mathrm{d}t}{\sqrt{a^2-t^2}} - \frac{\alpha}{a}\int_{-b}^{b}\frac{t^2\,\mathrm{d}t}{\sqrt{a^2-t^2}} - \alpha\int_{b}^{a}\frac{t\,\mathrm{d}t}{\sqrt{a^2-t^2}} \\ &= -2\frac{\alpha}{b}a^2\int_0^{\theta_0}\sin^2\theta\,\mathrm{d}\theta - 2\alpha a\int_{\theta_0}^{\pi/2}\sin\theta\,\mathrm{d}\theta \\ &= -\frac{\alpha a}{\sin\theta_0}\left(\theta_0 - \frac{\sin 2\theta_0}{2} + \sin 2\theta_0\right). \end{aligned} \tag{11}$$

We need to distinguish between two cases: (i) when the stick zone lies entirely within the rounded part of the indenter, i.e. $c < b$; (ii) when the stick zone extends into the linear part, i.e. $c > b$. In the first case (i):

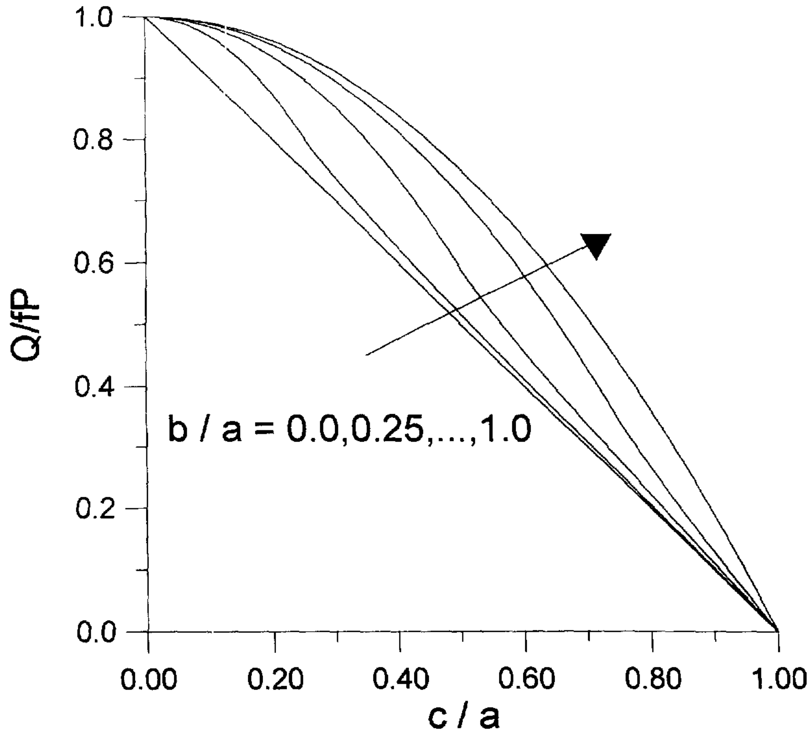

Fig. 2. Relationship $Q/fP - c/a$ for a wedge-shaped indenter with rounded apex, for different ratios $b/a = 0, 0.25, 0.5, 0.75, 1$, corresponding to the transition from sharp wedge to Hertzian indenter.

$$\int_{-c}^{c} \frac{h'(t)t\,\mathrm{d}t}{\sqrt{c^2-t^2}} = -\frac{\alpha}{b}\int_{-c}^{c} \frac{t^2\,\mathrm{d}t}{\sqrt{c^2-t^2}} = -2\frac{\alpha}{b}c^2\int_0^{\pi/2} \sin^2\theta\,\mathrm{d}\theta = -\frac{\alpha}{b}c^2\frac{\pi}{2} \tag{12}$$

and we find

$$\frac{Q}{fP} = 1 - \left(\frac{c}{a}\right)^2 \frac{\pi/2}{\theta_0 + \dfrac{\sin 2\theta_0}{2}}, \quad c < b. \tag{13}$$

In the second case (ii), we find that:

$$\frac{Q}{fP} = 1 - \left(\frac{c\sin\theta_0}{a\sin\omega_0}\right)\frac{\omega_0 + \dfrac{\sin 2\omega_0}{2}}{\theta_0 + \dfrac{\sin 2\theta_0}{2}} \quad c > b \tag{14}$$

where $b = c\sin\omega_0$.

The resulting behaviour in the partial slip regime is shown graphically in Fig. 2, for $b/a = 0, 0.25, 0.5, 0.75, 1$. It may be appreciated that the response is very smooth at the transition in the function profile. Indeed, we did expect a smooth transition from the linear to the parabolic behaviour, but clearly different from the one that can be obtained with a smooth transition of a power-law punch for k moving from $k = 1$ to $k = 2$.

As regarding the traction distribution, it may be shown that the pressure is given by

$$\frac{\pi A}{\alpha}p(\theta) = \ln\left|\tan\frac{\theta+\theta_0}{2}\tan\frac{\theta-\theta_0}{2}\right| - 2\theta_0\frac{\cos\theta}{\sin\theta_0} - \frac{\sin\theta}{\sin\theta_0}\ln\left|\frac{\sin(\theta-\theta_0)}{\sin(\theta+\theta_0)}\right| \tag{15}$$

where $-\pi/2 \leqslant \theta \leqslant \pi/2$, which corresponds to the physical region $-a \leqslant x = b\sin\theta/\sin\theta_0 \leqslant a$. The contact area dimension a is given by equilibrium

$$\frac{AP}{\alpha} = \frac{b}{\sin^2\theta_0}\left(\frac{\sin 2\theta_0}{2} + \theta_0\right). \tag{16}$$

In the partial slip regime, for $c < b$, the corrective shear $q^*(x)$ has the Cattaneo–Mindlin shape for a parabolic punch, whereas in the case $c > b$, mapping the region $-c \leqslant x \leqslant c$ by means of the relation $x = b\sin\omega/\sin\omega_0$, one has

$$-\frac{\pi A}{\alpha}q^*(\omega)/f = \ln\left|\tan\frac{\omega+\omega_0}{2}\tan\frac{\omega-\omega_0}{2}\right| - 2\omega_0\frac{\cos\omega}{\sin\omega_0} - \frac{\sin\omega}{\sin\omega_0}\ln\left|\frac{\sin(\omega-\omega_0)}{\sin(\omega+\omega_0)}\right|. \tag{17}$$

2.5. *Flat punch with rounded corners*

Consider a flat punch, with rounded corners, in contact with the half-plane. The function $h'(x)$ is described by

$$h'(x) = \begin{cases} -(b+x)/R, & -a \leqslant x \leqslant -b \\ 0, & -b \leqslant x \leqslant +b \\ -(x-b)/R, & +b \leqslant x \leqslant +a \end{cases} \tag{18}$$

so that

$$\begin{aligned} \int_{-c}^{c}\frac{h'(t)t\,\mathrm{d}t}{\sqrt{c^2-t^2}} &= -\frac{1}{R}\int_{-c}^{-b}\frac{(b+t)t\,\mathrm{d}t}{\sqrt{c^2-t^2}} - \frac{1}{R}\int_{b}^{c}\frac{(t-b)t\,\mathrm{d}t}{\sqrt{c^2-t^2}} \\ &= -\frac{2}{R}c^2\int_{\omega_0}^{\pi/2}\sin^2\theta\,\mathrm{d}\theta + 2\frac{bc}{R}\int_{\omega_0}^{\pi/2}\sin\theta\,\mathrm{d}\theta \\ &= -\frac{2}{R}c^2\left(\frac{\pi}{4} - \frac{\omega_0}{2} - \frac{\sin 2\omega_0}{4}\right) \end{aligned} \tag{19}$$

on substituting $t = c\sin\theta$, and defining $b = c\sin\omega_0$. In this configuration, only a partial slip case with stick zone greater than the flat central part is possible, i.e. $c > a$. The integral related to the entire contact area can be calculated from eqn (I-33), formally substituting $c = a$, i.e. θ_0 to ω_0, and a to c, obtaining

$$\frac{Q}{fP} = 1 - \left(\frac{c}{a}\right)^2\frac{\pi - 2\omega_0 - \sin 2\omega_0}{\pi - 2\theta_0 - \sin 2\theta_0}, \quad \frac{c}{a} > \frac{b}{a} \tag{20}$$

$$\frac{Q}{fP} = 1, \quad \frac{c}{a} < \frac{b}{a}. \tag{21}$$

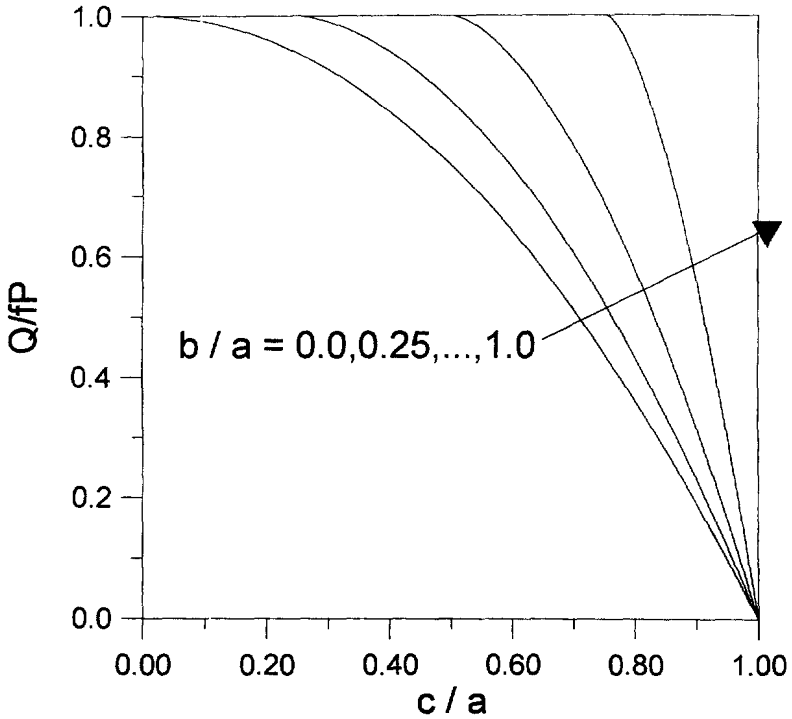

Fig. 3. Relationship $Q/fP - c/a$ for a flat indenter with rounded corners, for different ratios $b/a = 0, 0.25, 0.5, 0.75, 1$, corresponding to the transition from Hertzian indenter, to flat punch.

This result is plotted in Fig. 3, where it may be appreciated again that the transition from the Hertzian case ($b/a = 0$) to the limit flat punch is not smooth, as in the case of the transition through a polynomial punch $k = 2$ to $k \to \infty$. Moreover, by comparison with the previous case of the rounded wedge, there is more influence on the load application, as full sliding only occurs when the slip zone has reached the transition point in the profile. It is worth remarking, as a consequence, that wear cannot occur in the flat part of the indenter if the punch is in the partial slip regime.

As regards the traction distribution, let us start by considering the pressure. Mapping the physical contact region $-a \leqslant x \leqslant a$, ($a = b/\sin\theta_0$) by $x = b\sin\theta/\sin\theta_0$, the dimensionless pressure distribution is

$$\frac{bp(\theta)}{P} = -\frac{2/\pi}{\pi - 2\theta_0 - \sin 2\theta_0} \times \left\{ (\pi - 2\theta_0)\cos\theta + \ln\left[\left| \frac{\sin(\theta+\theta_0)}{\sin(\theta-\theta_0)} \right|^{\sin\theta} \left| \tan\frac{\theta+\theta_0}{2} \tan\frac{\theta-\theta_0}{2} \right|^{\sin\theta_0} \right] \right\}. \tag{22}$$

Considering b as a known geometrical quantity, the angle θ_0 is given by the load P as

$$\frac{PAR}{b^2} = \frac{\pi - 2\theta_0}{2\sin^2\theta_0} - \cot\theta_0. \tag{23}$$

Moving to partial slip regime ($c > b$), the dimensionless corrective shearing distribution is given by mapping the physical stick region $-c \leqslant x \leqslant c$, ($c = b/\sin\omega_0$) by $x = b\sin\omega/\sin\omega_0$:

$$-\frac{bq^*(\theta)}{Q^*}=-\frac{2/\pi}{\pi-2\omega_0-\sin 2\omega_0}$$
$$\times\left\{(\pi-2\omega_0)\cos\omega+\ln\left[\left|\frac{\sin(\omega+\omega_0)}{\sin(\omega-\omega_0)}\right|^{\sin\omega}\left|\tan\frac{\omega+\omega_0}{2}\tan\frac{\omega-\omega_0}{2}\right|^{\sin\omega_0}\right]\right\}. \quad (24)$$

2.6. *Truncated power-law punch*

Consider, now, a power-law punch with a central flat region This is an ideal vehicle to show that the relation Q/fP, size of stick zone, is much easier to calculate than the pressure distribution itself. In fact, we failed to obtain an analytical solution for the tractions. Consider a function $h'(x)$ described by

$$h'(x)=\begin{cases}Ck|x|^{k-1}, & -a\leqslant x\leqslant -b\\ 0, & -b\leqslant x\leqslant +b.\\ -Ck|x|^{k-1}, & +b\leqslant x\leqslant +a\end{cases} \quad (25)$$

Then

$$\int_{-a}^{a}\frac{h'(t)t\,\mathrm{d}t}{\sqrt{a^2-t^2}}=Ck\left[\int_{-a}^{-b}\frac{|t|^{k-1}t\,\mathrm{d}t}{\sqrt{a^2-t^2}}-\int_{b}^{a}\frac{|t|^{k-1}t\,\mathrm{d}t}{\sqrt{a^2-t^2}}\right]$$
$$=-2Ck\int_{b}^{a}\frac{t^k\,\mathrm{d}t}{\sqrt{a^2-t^2}}=-2Cka^k\int_{b/a}^{1}\frac{\tau^k\,\mathrm{d}\tau}{\sqrt{1-\tau^2}} \quad (26)$$

$$\int_{-c}^{c}\frac{h'(t)t\,\mathrm{d}t}{\sqrt{c^2-t^2}}=-2Ckc^k\int_{b/c}^{1}\frac{\tau^k\,\mathrm{d}\tau}{\sqrt{1-\tau^2}}. \quad (27)$$

Therefore, the stick zone, which must lie in the interval $c/a>b/a$, is given by

$$\frac{Q}{fP}=1-\left(\frac{c}{a}\right)^k\int_{b/c}^{1}\frac{\tau^k\,\mathrm{d}\tau}{\sqrt{1-\tau^2}}\Big/\int_{b/a}^{1}\frac{\tau^k\,\mathrm{d}\tau}{\sqrt{1-\tau^2}},\quad \frac{c}{a}>\frac{b}{a} \quad (28)$$

$$\frac{Q}{fP}=1,\quad \frac{c}{a}<\frac{b}{a} \quad (29)$$

where the integrals can be computed in terms of special functions (details are given in Appendix 2).

2.7. *Spline profile*

Lastly, in this section of symmetrical profiles, consider a profile such that the function $h'(x)$ is defined in the generic region $[-b,-a]\cup[a,b]$ by

$$h'(x)=mx+D,\quad -b\leqslant x\leqslant -a$$
$$h'(x)=mx-D,\quad +a\leqslant x\leqslant +b \quad (30)$$

where

$$m=-\frac{h'(b)-h'(a)}{b-a},\quad D=h'(a)+ma. \quad (31)$$

On defining d, the dimension of the contact area, $\sin\theta = x/d$, $\sin\theta_a = a/d$, $\sin\theta_b = b/d$, considering a case with n sections of the profile, it may be proved (Ciavarella *et al.*, 1997c) the solution for the pressure is given by

$$p(\theta) = \frac{1}{\pi A}\sum_{i=0}^{n-1}\left[2m_i d(\theta_{i+1}-\theta_i)\cos\theta + m_i d\sin\theta\ln\left|\frac{\sin(\theta+\theta_i)}{\sin(\theta+\theta_{i+1})}\frac{\sin(\theta-\theta_{i+1})}{\sin(\theta-\theta_i)}\right| + D_i\ln\left|\frac{(\cos\theta-\cos\theta_i)(\cos\theta+\cos\theta_{i+1})}{(\cos\theta-\cos\theta_{i+1})(\cos\theta+\cos\theta_i)}\right|\right] \quad (32)$$

and the total load is

$$\frac{P}{d} = \frac{1}{A}\sum_{i=0}^{n-1}\left[2D_i(\cos\theta_i-\cos\theta_{i+1}) - \frac{m_i d}{2}(\sin 2\theta_i - \sin 2\theta_{i+1}) + m_i d(\theta_i-\theta_{i+1})\right] \quad (33)$$

where $\theta_n = \pi/2$, as it corresponds to the contact area edge.

In the particular case there is a flat central area, $[-a_0, a_0]$, the summation has to start from the corresponding angle θ_0. Then, the above result continue to apply, but expression refers to $n+1$ sections.

On moving to the partial slip case, define c the dimension of the contact area, by $\sin\omega = x/c$, $\sin\omega_a = a/c$, $\sin\omega_b = b/c$, considering a case with $n^* < n$ sections of the profile that are in the stick zone, the corrective shear is given by

$$-q^*(x)/f = \frac{1}{\pi A}\sum_{i=0}^{n^*-1}\left[2m_i c(\omega_{i+1}-\omega_i)\cos\omega + m_i c\sin\omega\ln\left|\frac{\sin(\omega+\omega_i)}{\sin(\omega+\omega_{i+1})}\frac{\sin(\omega-\omega_{i+1})}{\sin(\omega-\omega_i)}\right| + D_i\ln\left|\frac{(\cos\omega-\cos\omega_i)(\cos\omega+\cos\omega_{i+1})}{(\cos\omega-\cos\omega_{i+1})(\cos\omega+\cos\omega_i)}\right|\right] \quad (34)$$

and the corrective load is given by

$$-\frac{Q^*}{c}/f = \frac{1}{A}\sum_{i=0}^{n^*-1}\left[2D_i(\cos\omega_i-\cos\omega_{i+1}) - \frac{m_i c}{2}(\sin 2\omega_i - \sin 2\omega_{i+1}) + m_i c(\omega_i-\omega_{i+1})\right] \quad (35)$$

where $\omega_n = \pi/2$, as it corresponds to the stick area edge.

2.8. *A non-symmetrical wedge*

To present an example of the treatment of non-symmetrical contacts, where the centre of the contact area is unknown *a priori*, we summarize here results for the simplest case of a wedge non-symmetrical with respect to y-axes. A full account of the derivation of the results below, and more general treatment of non-symmetrical contact, can be found in the paper (Ciavarella and Demelio, 1997).

Let us consider a case of a non-symmetrical wedge where the geometry is such that

$$h'(x) = \begin{cases} -k_-, & x \leqslant 0 \\ -k_+, & 0 \leqslant x \end{cases} \quad (36)$$

where k is related to the external angle of the wedge, i.e. $k_+ = \theta$, $k_- = -\theta$ for a symmetrical wedge. Note that the case with rotation is readily incorporated into the value of k_+, k_-.

Now, define

$$x = \xi + \delta, \quad \xi = a\cos\phi, \quad \delta = -a\cos\phi_0 \tag{37}$$

we find

$$p(\phi) = -\frac{a}{\pi A}(k_+ - k_-)\ln\left|\frac{\sin\dfrac{\phi+\phi_0}{2}}{\sin\dfrac{\phi-\phi_0}{2}}\right|. \tag{38}$$

The last two formulas determine $p(x)$ in the range $-a+\delta < x < a+\delta$.

Substituting now in the conditions for $p(x)$ at the edge to be zero, and using $\tau = a\cos\phi$, $\delta = -a\cos\phi_0$, we find

$$\phi_0 = \frac{\pi k_-}{k_- - k_+}, \quad a = \frac{AP}{(k_+ - k_-)\sin\dfrac{\pi k_-}{k_- - k_+}}. \tag{39}$$

Substituting in the relation for M, we obtain

$$M = P\delta + \frac{P}{2}a\cos\phi_0 = \frac{1}{2}P\delta \tag{40}$$

since $a\cos\phi_0 = -\delta$. Thus, for equilibrium it is necessary that the resultant P is displaced of $x = 1/2\delta$ if no rotation has to occur. Clearly, in the limit $\delta = a$, one contact edge reaches the apex of the wedge, but this limit case is not meaningful, as it may be shown to correspond to a flat punch with a linear profile on one side, so that the other edge is undefined. Therefore, the condition $k_+ > 0$, $k_- < 0$ has to be added for the solution to be meaningful.

On moving to the partial slip regime, let us define

$$x = \eta + \delta^*, \quad \eta = c\cos\omega, \quad \delta^* = -\cos\omega_0 \tag{41}$$

we find

$$-q^*(x)/f = -\frac{c}{\pi A}(k_+ - k_-)\ln\left|\frac{\sin\dfrac{\omega+\omega_0}{2}}{\sin\dfrac{\omega-\omega_0}{2}}\right|. \tag{42}$$

The last two formulae determine $q^*(x)$ in the range $-c+\delta^* < x < c+\delta^*$.

Substituting now in the conditions for $q^*(x)$ at the edge of the stick area to be zero, one obtains

$$\omega_0 = \frac{\pi k_-}{k_- - k_+}, \quad c = \frac{AQ^*/f}{(k_+ - k_-)\sin\dfrac{\pi k_-}{k_- - k_+}}. \tag{43}$$

The fact that $\omega_0 = \phi_0$ was expected, as the profile is self-similar, so that $\delta^*/\delta = c/a$.

3. MULTIPLE CONTACTS

On moving to the case where the contact area is composed by a (finite) number of regions of the x-axes, i.e. the case of a multiply connected contact area, the solution becomes immediately more complicated. Gladwell (1980) presents a solution for the case of two parabolic punches, whereas Schtayerman (1949) gives solutions to the cases of two flat areas of different height. Here, we recollect the latter solution, whereas for the general case, the use of the analytical solution, given in part I of the paper (specifically, Appendix 1) becomes quite cumbersome. Therefore, as perhaps an approximate numerical scheme is satisfactory for a number of cases, we present a simple numerical scheme considering the interaction between different areas in a simplified first-order approximation.

3.1. *Two flat areas*

If the profile is such that h is constant over two areas, say of equal size, and consider the case when the punch is brought into contact by a rigid vertical body motion, so that

$$h(x) = \begin{cases} T_y, & -b \leqslant x \leqslant -a \\ T_y+\gamma, & +a \leqslant x \leqslant +b \end{cases}. \tag{44}$$

The solution, when the contact is extended to both areas, is (Scthayerman, 1949)

$$p(x) = \mp \frac{C_0 - Px}{\pi\sqrt{(x^2-a^2)(b^2-x^2)}}, \quad \begin{cases} -b \leqslant x \leqslant -a \\ +a \leqslant x \leqslant +b \end{cases} \tag{45}$$

where

$$C_0 = -\frac{\pi b \gamma}{2AK(k)}, \quad k = a/b \tag{46}$$

and $K(k)$ is the elliptic integral of the first kind. The pressure is clearly higher in the lower surface. During normal loading, two different phases can be detected. At first, only the lower surface is in contact, and the pressure is given by the flat punch case, eqn (4); when the upper surface comes into contact, the pressure is given by eqn (46), and it is clearly not symmetrical anymore (eqn (1-9) can be applied to calculate the moment). This corresponds, in the Cattaneo partial slip problem, to the fact that there are two phases before full sliding: at first application of the tangential load, there is full stick; then, as Q^*/f reaches the value corresponding to the normal load when the upper surface is in incipient contact, there is full slip on that surface, whereas full-stick continues in the lower surface. Only at $|Q| = fP$ does the lower surface slip suddenly.

4. PERIODIC CONTACTS

The case of periodic contact is of great engineering interest, in that it encompasses the characteristics of rough contacts, as long as it can be assumed that the roughness has some sort of deterministic periodicity (in the more general case, still a Fourier analysis of the roughness gives a spectrum of heights of the asperities as a function of the wavelengths). Therefore, the elementary case of a sine profile gives an insight into the simplest way in on this problem. The partial slip solution is here provided as an extension, and indeed this permits us to achieve a better understanding of the partial slip regime for these class of contacts.

4.1. *Sine wave profile*

When an infinite number of punches acts on the half-plane, it is impossible to consider total normal or tangential forces, and it is even naïve (in plane problems) to compute the rigid body motion. We may, therefore, consider only the distribution of tangential traction, and the average pressure (equivalently, we can define the total load per contact area). For a profile in the form of a cosine wave

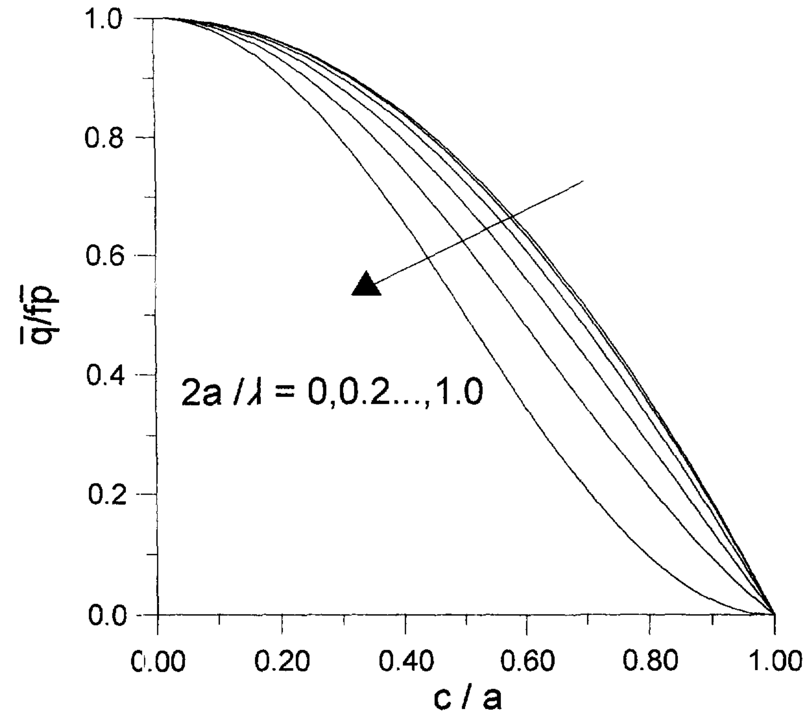

Fig. 4. Relationship $\bar{q}/f\bar{p}-c/a$ for a periodic contact, for different level of load, i.e. $2a/\lambda = 0.2, 0.4, 0.6, 0.8, 1$, corresponding to the transition from far interaction, to complete contact of the half-planes.

$$f_1(x) = \Delta\cos\left(\frac{2\pi x}{\lambda}\right) \tag{47}$$

in contact with a half-plane ($f_2(x) = 0$), Westergaard's solution applies [see Hills *et al.*, (1993) Section 14.2, for more details on the derivation], with the outcome

$$p(x) = \frac{-2\bar{p}}{\sin^2(\pi a/\lambda)}[\sin^2(\pi a/\lambda)-\sin^2(\pi x/\lambda)]^{1/2} \tag{48}$$

applied over the regions $|x-n\lambda| < a$, with $p(x) = 0$ outside these regions, and

$$\bar{p} = 2\pi\Delta\sin^2(\pi a/\lambda)/A\lambda. \tag{49}$$

This permits the determination of the size of the contact area a. For light loads ($2a/\lambda$ less than about 0.4) a Hertzian contact pressure distribution results at each asperity whose curvature we may write as $k = 4\pi^2\Delta/\lambda^2$, and the interaction is negligible. The other limiting case, of an extremely heavy load, $2a/\lambda = 1$, gives a sinusoidally varying continuous pressure distribution. These formulae can be directly processed to give the solution to Cattaneo's partial slip problem. The average pressure $\bar{p}$ transforms into an average corrective shearing traction distribution $\bar{q}^*$, which determines the size c of the stick zone, and

$$\frac{\bar{q}^*}{f\bar{p}} = 1-\left(\frac{\sin(\pi c/\lambda)}{\sin(\pi a/\lambda)}\right)^2 \tag{50}$$

and the shearing tractions are given by

$$q(x) = \begin{cases} fp(x) + \dfrac{2\bar{a}^*}{\sin^2(\pi c/\lambda)}[\sin^2(\pi c/\lambda)-\sin^2(\pi x/\lambda)]^{1/2} & x \in S_{\text{stick}} \\ fp(x) & x \in S_{\text{slip}} \end{cases}. \tag{51}$$

Figure 4 summarizes the results obtained for the ratios $2a/\lambda = 0.2, 0.4, 0.6, 0.8, 1.0$, i.e.

from a practically Hertzian case (no interaction due to contact wide apart), to the complete contact of the half-planes. It is interesting to note that the variation is not particularly great, as compared to the variation due to different profiles that we have discussed so far. Of course, the effect here is interaction, and if the profiles are different from a sinusoidal profile, the combined effects may sum, although it is not possible in general to achieve analytical results. Note that in the case where the load is increased to more than what is needed to full contact, there is a uniform pressure to add to eqn (48). In this case, on applying a tangential load, it is clear that full stick will occur at first, and only when the tangential load has reached the point corresponding to the loose of full contact, the partial slip condition hold.

4.2. *Flat periodic contact*

Here the solution corresponds to the frictionless normal contact in the case of a periodical set of flat surfaces of equal height, i.e. $h'(x) = 0$, over the range $a_m \leqslant x \leqslant b_m$, $(m = 1, \ldots, n)$, where the period is equal to only one contact area, $n = 1$, $h'(x) = 0$, over $l/2 - a \leqslant x \leqslant l/2 + a$. The solution is

$$p\left(\frac{l}{2}+x\right) = -\frac{P\sqrt{2}\cos\dfrac{\pi x}{l}}{l\sqrt{\cos\dfrac{\pi x}{l} - \cos\dfrac{\pi a}{l}}}, \quad -a \leqslant x \leqslant a \tag{52}$$

where P is the load per contact area. Note that in the limit when $l = 2a$, the shearing distribution goes towards the uniform limit.

Moving to the partial slip configuration, as the punch is composed of a infinite number of flat areas of equal heights, then a partial slip solution is not possible, from the general results already obtained. The contact will be either in full stick, or full slip regime, and the shearing will be always proportional to the pressure as given above, i.e.

$$q\left(\frac{l}{2}+x\right) = -\frac{Q\sqrt{2}\cos\dfrac{\pi x}{l}}{l\sqrt{\cos\dfrac{\pi x}{l} - \cos\dfrac{\pi a}{l}}}, \quad -a \leqslant x \leqslant a \tag{53}$$

where Q is the total tangential load per contact area. As soon as $|Q| = fP$, the full slip conditions arise simultaneously in the entire contact area.

4.3. *Sine squared wave profile*

For a profile in the form of a squared cosine wave

$$f_1(x) = \Delta\cos^2\left(\frac{\pi x}{\lambda}\right) \tag{54}$$

in contact with a half-plane ($f_2(x) = 0$), Scthayerman (1949) gives the solution

$$p\left(\frac{\lambda}{2}+x\right) = \sqrt{2}\frac{\lambda}{\lambda_0}\frac{P}{\lambda_0}\cos\left(\frac{\pi x}{\lambda}\right)\left[\cos\left(\frac{2\pi x}{\lambda}\right) - \cos\left(\frac{2\pi a}{\lambda}\right)\right]^{1/2}, \quad -a < x < a \tag{55}$$

where

$$\frac{a}{\lambda} = \frac{1}{\pi}\arcsin\frac{\lambda_0}{\lambda} \tag{56}$$

and

$$\frac{\lambda_0}{\lambda} = \sqrt{\frac{\Delta PA}{\pi}}. \tag{57}$$

When $\lambda_0/\lambda = 1$, then there is full contact. Moving to the partial slip case, as usual the solution is obtained using a corrective shearing distribution equal to

$$-q^*\left(\frac{\lambda}{2}+x\right) = \sqrt{2}\frac{\lambda}{\lambda_0}\frac{Q^*}{\lambda_0}\cos\left(\frac{\pi x}{\lambda}\right)\left[\cos\left(\frac{2\pi x}{\lambda}\right)-\cos\left(\frac{2\pi c}{\lambda}\right)\right]^{1/2}, \quad -c < x < c \tag{58}$$

where

$$\frac{c}{\lambda} = \frac{1}{\pi}\arcsin\frac{\lambda_0^*}{\lambda} \tag{59}$$

and

$$\frac{\lambda_0^*}{\lambda} = \sqrt{\frac{\Delta Q^* A}{\pi f}}. \tag{60}$$

5. CONCLUSIONS

Cattaneo's partial slip contact has been considered and new solutions developed, using the method proposed in Part I of the paper. The range of analytical results covers a vast range of configurations of engineering interest. For single contact area: punches of polynomial or power-law profile, wedge with rounded apex, flat punch with rounded corners and, more generally, a spline approximation of the profile; finally a truncated power-law punch. For multiple contacts, the case of two flat areas of different heights is treated. For periodic contacts: a sine-wave, a sine squared, and periodic flat profiles have been considered.

To treat more general contact problems, the results of part I of the paper permit to solve the Cattaneo partial problem as a frictionless normal contact: therefore, either the solutions in quadrature given in Part I of the paper has to be solved numerically, or a more classical numerical technique to solve frictionless normal contact has to be used.

Acknowledgements—The author is indebted to Professor J.R. Barber, of University of Michigan (U.S.A.), and Dr D.A. Hills, of University of Oxford, and Lincoln College, Oxford, U.K., and one referee for their suggestions.

REFERENCES

Carter, F. W. (1926). On the action of a locomotive driving wheel. *Proceedings of the Royal Society, Series A* **112,** 151–157.

Cattaneo, C. (1938). Sul contatto di due corpi elastici: distribuzione locale degli sforzi. *Rendiconti dell'Accademia Nazionale dei Lincei* **27,** 342–348, 434–436, 474–478.

Ciavarella, M. and Demelio, G. (1997). *On Non-Symmetrical Plane Contacts* (in preparation).

Ciavarella, M., Hills, D. A. and Monno, G. (1997a). Contact problems for a wedge with rounded apex. *International Journal of Mechanical Science* (submitted).

Ciavarella, M., Hills, D. A. and Monno, G. (1997b). Influence of rounded corners on identation by flat punches. *International Journal of Engineering Science* (submitted).

Ciavarella, M., Demelio, G. and Monno, G. (1997c). *A General Solution for Plane Contact of Spline Profiles* (in preparation).

Gladwell, G. M. L. (1980). *Contact Problems in the Classical Theory of Elasticity*. Sijthoff and Noordhoff, The Netherlands.

Hills, D. A., Nowell, D. and Sackfield, A. (1993). *Mechanics of Elastic Contacts.* Butterworth–Heinemann, Oxford.
Scthayerman, I. Ya. (1949). *Contact Problem of the Theory of Elasticity.* Gostekhteoretizdat, Moscow (available from the British Library in a English translation by Foreign Technology Div., FTD-MT-24-61-70, 1970).
Truman, C. E., Sackfield, A. and Hills, D. A. (1995). Contact mechanics of wedge and cone indenters. *International Journal of Mechanical Science* **37**(3), 261–275.

APPENDIX 1

Solution for a punch of profile x^k

Consider a punch of power-law profile $f_1(x) = C|x|^k$, including polynomial or any order, in contact with a half-plane. The function $h(x)$ is defined by

$$h'(x) = -\mathrm{sign}(x)Ck|x|^{k-1}. \tag{A1}$$

Then, the pressure can be obtained as

$$p(x) = \frac{Ck}{\pi A}\sqrt{a^2-x^2}\left[\int_{-a}^{0}\frac{|t|^{k-1}\,\mathrm{d}t}{\sqrt{a^2-t^2}(t-x)} - \int_{0}^{a}\frac{|t|^{k-1}\,\mathrm{d}t}{\sqrt{a^2-t^2}(t-x)}\right] \tag{A2}$$

$$= \frac{Ck}{\pi A}\sqrt{a^2-x^2}\left[\int_{-a}^{0}\frac{|t|^{k-1}(t+x)\,\mathrm{d}t}{\sqrt{a^2-t^2}(t^2-x^2)} - \int_{0}^{a}\frac{|t|^{k-1}(t+x)\,\mathrm{d}t}{\sqrt{a^2-t^2}(t^2-x^2)}\right] \tag{A3}$$

$$= -2\frac{Ck}{\pi A}\sqrt{a^2-x^2}\left[\int_{0}^{a}\frac{t^k\,\mathrm{d}t}{\sqrt{a^2-t^2}(t^2-x^2)}\right]. \tag{A4}$$

To calculate the normal load

$$P = -\frac{Ck}{A}\left[\int_{-a}^{0}\frac{|t|^{k-1}t\,\mathrm{d}t}{\sqrt{a^2-t^2}} - \int_{0}^{a}\frac{|t|^{k-1}t\,\mathrm{d}t}{\sqrt{a^2-t^2}}\right] \tag{A5}$$

$$= 2\frac{Ck}{A}\int_{0}^{a}\frac{t^k\,\mathrm{d}t}{\sqrt{a^2-t^2}}. \tag{A6}$$

Substituting $x = as$, and $t = a\tau$, and using the result†

$$\int_0^1 \frac{\tau^k\,\mathrm{d}\tau}{\sqrt{1-\tau^2}(\tau^2-s^2)} = \frac{\sqrt{\pi}}{2}\left[\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\,{}_2F_1\left(1,1-\frac{k}{2};\frac{3-k}{2};s^2\right) + \frac{s^{k-1}}{\sqrt{1-s^2}}\tan\left(k\frac{\pi}{2}\right)\right] \tag{A7}$$

$$\int_0^1 \frac{\tau^k\,\mathrm{d}\tau}{\sqrt{1-\tau^2}} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)} \tag{A8}$$

one has for $|s| < 1$,

$$p(s) = -2\frac{Ck}{\pi A}a\frac{\sqrt{\pi}}{2}\left[\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\,{}_2F_1\left(1,1-\frac{k}{2};\frac{3-k}{2};s^2\right)\sqrt{1-s^2} + s^{k-1}\tan\left(k\frac{\pi}{2}\right)\right] \tag{A9}$$

where ${}_2F_1(.,.;.;z)$ is the Gauss hypergeometric function of argument z, and

† Mathematica Vol. 3.0, Wolfram Research, Champain, Illinois, U.S.A.

$$P = 2\frac{Ck}{A}a^k\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)}. \tag{A10}$$

For particular values of k, simplified results are possible, and in particular it is evident that for $k = 2$ the classical Hertzian parabolic case is recovered.

APPENDIX 2

Truncated power-law punch

We note the results†,

$$\int_z^1 \frac{\tau^k \, d\tau}{\sqrt{1-\tau^2}} = \frac{\sqrt{\pi}}{2} - \frac{1}{1+k} z^{1+k} {}_2F_1\left(\frac{1}{2}, \frac{1+k}{2}; \frac{3+k}{2}; z^2\right)$$

where ${}_2F_1$ is the Gauss hypergeometric function. From there, the values of relation normal load–contact area, or tangential load–stick zone can be calculated in partial slip (as usual, only the case for which $c/a > b/a$ can be considered)

$$\frac{Q}{fP} = 1 - \left(\frac{c}{a}\right)^k \frac{\frac{\sqrt{\pi}}{2} - \frac{1}{1+k}\left(\frac{b/a}{c/a}\right)^{1+k} {}_2F_1\left(\frac{1}{2}, \frac{1+k}{2}; \frac{3+k}{2}; \left(\frac{b/a}{c/a}\right)^2\right)}{\frac{\sqrt{\pi}}{2} - \frac{1}{1+k}\left(\frac{b}{a}\right)^{1+k} {}_2F_1\left(\frac{1}{2}, \frac{1+k}{2}; \frac{3+k}{2}; \left(\frac{b}{a}\right)^2\right)}, \quad \frac{c}{a} > \frac{b}{a} \tag{A11}$$

whereas $Q/fP = 1, c \leqslant b$. To remark the case of a truncated parabolic punch, that can arise for example on a Hertzian punch, whose profile has been flattened in the middle for example by wear. In that case, it is possible to use the result

$${}_2F_1\left(\frac{1}{2}, \frac{3}{2}; \frac{5}{2}; z\right) = -\frac{3}{2z^{3/2}}[\sqrt{z}\sqrt{1-z} - \arcsin\sqrt{z}]. \tag{A12}$$

† Mathematica Vol. 3.0, Wolfram Research, Champain, Illinois, U.S.A.